On uniform Roe algebras of locally finite groups

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Groups viewed as metric spaces

- G - countable discrete group
- S - finite generating set such that $e \in S$ and $S^{-1} = S$
- $d(g, h) =$ smallest number of generators connecting g and h

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Example

- $G = \mathbb{Z}$, $S = \{-1, 0, 1\}$. Then $d(m, n) = |m - n|$.
- $G = \mathbb{Z}^d$, $S = \{\pm \mathbf{e}_i : i = 1, 2, ..., d\} \cup \{\mathbf{0}\}$. Then $d(\mathbf{m}, \mathbf{n}) = \sum_{i=1}^{d} |m_i - n_i|$

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Question

How does the metric depend on the generating set?

Coarse equivalence

Definition

A map $f : (X, d_X) \to (Y, d_Y)$ is **uniformly expansive** if for any $A > 0$ there exists $B > 0$ such that $d_X(x, y) < A$ implies $d_Y(f(x), f(y)) < B$ for all x, y.

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A uniformly expansive map $f : X \to Y$ is a **coarse equivalence** if there exists a uniformly expansive map $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

Examples

Example

- The inclusion map $\iota : \mathbb{Z} \to \mathbb{R}$ is a coarse equivalence. A "coarse inverse" of ι is given by the floor map $x \mapsto \lfloor x \rfloor$.
- Any map between two bounded spaces is a coarse equivalence.
- Let H be a finite index subgroup of G. Then H and G are coarsely equivalent.
- Let G be a finitely generated group. Then G is coarsely equivalent to its Cayley graph.

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Non-example

The set $\{\pm n^2 : n \in \mathbb{N}\} = \{..., -16, -9, -4, -1, 0, 1, 4, 9, 16, ...\}$ is **not** coarsely equivalent to $\mathbb{Z}$.

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In fact, everything works for groups which are **not** finitely generated:

Theorem (Tu)

Let G be a countable group. Then there exists a proper left-invariant metric d on G, and it is unique up to (bijective) coarse equivalence.

Uniform Roe algebras

Let (X, d) be a countable metric space.

Definition

An operator T in $B(\ell^2(X))$ is said to have **finite propagation** if there exists $S > 0$ such that $\langle T\delta_y, \delta_x \rangle = 0$ whenever $d(x, y) \geq S$.

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- $\mathbb{C}_u^*[X]$ - the collection of all operators with finite propagation.
- $C_u^*(X)$ - the norm closure of $\mathbb{C}_u^*[X]$ (**uniform Roe algebra**)

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Example

When X is finite, $C_u^*(X) = B(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$.

C^*-algebraic properties $\iff$ geometric properties

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The converse holds in many cases.

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Theorem (Rørdam-Sierakowski)

$C_u^*(G)$ *is properly infinite* $\iff$ G *is non-amenable.*

Locally finite groups

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Example

- Every finite group is locally finite.
- An infinite direct sum of finite groups, such as $\bigoplus_{i=0}^{\infty} \mathbb{Z}/2\mathbb{Z}$, is locally finite
- $S_\infty = \{\text{finitary permutation of } \mathbb{N}\}$ is locally finite.
- $\mathbb{Q}/\mathbb{Z}$ is locally finite.

Asymptotic dimension

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- If X is bounded, then $\mathrm{asdim}(X) = 0$.
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Theorem (Smith)

Let G be a countable group. Then G is locally finite if and only if $\mathrm{asdim}(G) = 0$.

C^*-characterizations of local finiteness

We say a C^*-algebra A is **locally finite-dimensional** if for any finite subset $F \subseteq A$ and $\varepsilon > 0$ there exists a finite-dimensional subalgebra B such that $d(x, B) < \varepsilon$ for all $x \in F$.

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Theorem (Winter-Zacharias, Scarparo)

Let G be a countable group. Then the following are equivalent:

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- (1) $\implies$ (2): Winter-Zacharias on nuclear dimension.
- (3) $\implies$ (1): Scarparo's work on locally finite actions.

K-theory

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K-theory encodes everything of separable locally finite-dimensional C^*-algebras (Elliott's classification theorem). How much information does it carry in the case of $C_u^*(G)$?

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Theorem (Li-L.)

Let G and H be countable locally finite groups. Then the following are equivalent:

1. *G and H are bijectively coarsely equivalent;*
2. *$C_u^*(G)$ and $C_u^*(H)$ are $*$-isomorphic;*
3. *$K_0(C_u^*(G))$ and $K_0(C_u^*(H))$ are isomorphic as ordered abelian groups with distinguished units.*